Tutorial 1 (Jan 15, 17)

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Recall the definition of volume beneath a surface:

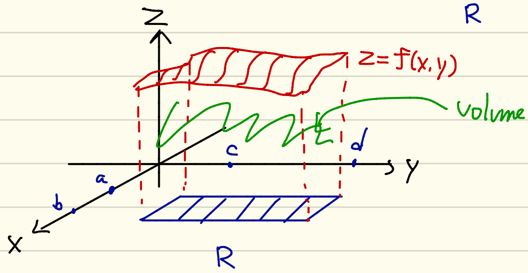
Def Let $R=[a,b]\times[c,d]$ be a rectangle in $\mathbb{R}^2$,

$f: R \to \mathbb{R}$ be a positive continuous function.

Volume of the region bounded above by graph of f and below by R

is defined to be the double integral $\iint\limits_R f(x,y)\,dA$

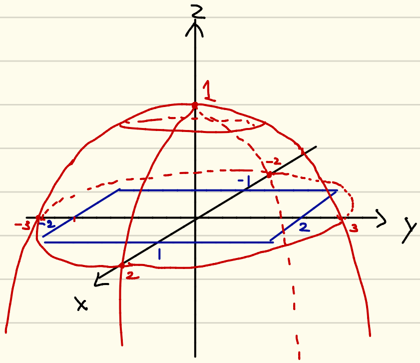
Picture:



Q1) Find the volume of the region bounded above by the elliptic paraboloid

$z = 1 - \frac{x^2}{4} - \frac{y^2}{9}$ and below by the rectangle $R = [-1,1] \times [-2,2]$

Picture:



Sol: Define $f : R \to \mathbb{R}$ by $f(x,y) = 1 - \frac{x^2}{4} - \frac{y^2}{9}$, then

$$\text{Volume} = \iint_R f(x,y)\,dA = \int_{-1}^{1}\int_{-2}^{2}\left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right)dy\,dx \quad \text{(by Fubini's Theorem)}$$

$$= 4\int_0^1\int_0^2\left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right)dy\,dx \quad \left(\begin{array}{l}\text{as } f(x,y) \text{ is an even function} \\ \text{with respect to } x, y\end{array}\right)$$

$$= 4\int_0^1\left[\left(1 - \frac{x^2}{4}\right)y - \frac{y^3}{27}\right]_0^2 dx$$

$$= 4\int_0^1\left(2\left(1 - \frac{x^2}{4}\right) - \frac{8}{27}\right)dx$$

$$= 4\int_0^1\left(-\frac{x^2}{2} + \frac{46}{27}\right)dx$$

$$= 4\cdot\left[-\frac{x^3}{6} + \frac{46}{27}x\right]_0^1 = 4\cdot\left(-\frac{1}{6} + \frac{46}{27}\right) = \frac{166}{27}$$

Q2) Evaluate $\iint_R f(x,y)\,dA$, where $R=[0,1]\times[-3,3]$; $f(x,y)=\dfrac{xy^2}{x^2+1}$

Sol Method 1: Using Fubini's Theorem:

$$\iint_R f(x,y)\,dA = \int_0^1\int_{-3}^{3}\frac{xy^2}{x^2+1}\,dy\,dx$$

$$=\int_0^1\left[\left(\frac{x}{x^2+1}\right)\left(\frac{y^3}{3}\right)\right]_{-3}^{3}dx$$

$$=\int_0^1\frac{x}{x^2+1}\cdot 18\,dx$$

$$=9\cdot\int_0^1\frac{d(x^2+1)}{x^2+1} = 9\cdot\left[\log(x^2+1)\right]_0^1 = 9\log 2$$

Method 2: Using the "separability" of $f(x,y)$

i.e. $f(x,y)=g(x)h(y)$, where $\begin{cases} g(x)=\dfrac{x}{x^2+1} : [0,1]\to\mathbb{R} \\ h(y)=y^2 : [-3,3]\to\mathbb{R}\end{cases}$

then $$\iint_R f(x,y)\,dA = \int_0^1\int_{-3}^{3} g(x)h(y)\,dy\,dx$$

$$=\int_0^1 g(x)\left(\int_{-3}^{3}h(y)\,dy\right)dx$$

$$=\left(\int_0^1 g(x)\,dx\right)\left(\int_{-3}^{3}h(y)\,dy\right)=(\log 2)\cdot 9 = 9\log 2$$

Q3) Evaluate $\int_0^2 \int_0^3 y e^{-xy} \, dy \, dx$

Sol First attempt: Direct computation

$$\int_0^2 \int_0^3 y e^{-xy} \, dy \, dx = \int_0^2 \left(\int_0^3 y \cdot \underbrace{\left(-\frac{1}{x}\right)}_{\text{NOT defined at } x=0!} d(e^{-xy}) \right) dx$$

Correct attempt: Using Fubini's Theorem:

$$\int_0^2 \int_0^3 y e^{-xy} \, dy \, dx = \int_0^3 \int_0^2 y e^{-xy} \, dx \, dy$$

$$= \int_0^3 \left(\int_0^2 e^{-xy} \, d(xy) \right) dy$$

$$= \int_0^3 \left[-e^{-xy} \right]_0^2 dy$$

$$= \int_0^3 \left(-e^{-2y} + 1 \right) dy$$

$$= \left[\frac{e^{-2y}}{2} + y \right]_0^3$$

$$= \left(\frac{e^{-6}}{2} + 3 \right) - \frac{1}{2} = \frac{e^{-6} + 5}{2}$$